TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 265, Number 2, June 1981

MULTIPLICATIVELY INVARIANT SUBSPACES OF BESOV SPACES

BY

PER NILSSON

ABSTRACT. We study subspaces of Besov spaces $\dot{B}_p^{s,q}$ which are invariant under pointwise multiplication by characters. The case $s > 0$ is completely described, and for the case $s < 0$ we extend known results.

Multiplicatively invariant subspaces of Besov spaces. The study of subspaces of the homogeneous Besov spaces $\dot{B}_p^{s,q}$ invariant for multiplication by characters was initiated by R. Johnson [4], [5]. In this paper we will simplify and improve some of his results and we will also settle some points left open in [4]. More generally if X is any Banach space of (tempered) distributions we associate with X several multiplicative invariant subspaces $\pi \cdot X$, $\pi_c X$, $\pi_{\omega(h)} X$, $\pi_s X = \pi_{(1+|h|^2)^{s/2}} X$. In the case of Besov spaces with $s > 0$ we find that $\pi_s \dot{B}_p^{s,q} = \pi \cdot \dot{B}_p^{s,q}$ (which is Theorem 1.1 in [4]) while $\pi_{s-\varepsilon} \dot{B}_p^{s,q} = 0$ if $\varepsilon > 0$. It turns out that the cases $s = 0$ and $s < 0$ behave quite differently. For instance if $s < 0$ we prove $\pi_c \dot{B}_p^{s,q} = 0$ if $s < -n/p'$ or $s = n/p'$, $q = \infty$. In [4] this is done only for the case $1 \leqslant p \leqslant 2$. We also remark that these techniques have applications to Fourier multipliers between L^p-spaces. See [4], [5].

While in [4] the Besov spaces are defined using finite differences we will use the definition of Besov spaces depending on a general partition on the Fourier side as in [8]. From this definition it is apparent that the origin on the Fourier side plays a particular role. Since multiplication by a character means translation after Fourier transform this puts a heavy restriction on the multipliers.

I wish to express my gratitude to Professor J. Peetre for his guidance during my work on this paper.

0. Conventions. All function or distribution spaces are considered on $\mathbf{R}^n$. Likewise all integrals without integration limits are taken over all $\mathbf{R}^n$. In particular, L^p, where $1 \leqslant p \leqslant \infty$, denotes the Lebesgue space of measurable functions f such that the norm $\|f\|_p = (\int |f|^p \, dx)^{1/p}$ is finite. We let $\mathfrak{M}$ be the space of bounded measures on $\mathbf{R}^n$ and denote its norm likewise by $\| \cdot \|_1$. As usual $\mathcal{S}$ is the space of rapidly decreasing functions and its dual, the space of tempered distributions, will be denoted by $\mathcal{S}'$. Similarly $\mathfrak{D}'$ is the space of all (L. Schwartz) distributions. The relation $X \subset Y$, where X and Y are topological vector spaces, means that we have

Received by the editors March 31, 1980.
1980 *Mathematics Subject Classification*. Primary 46E35; Secondary 46F05.
Key words and phrases. Besov spaces, Morrey spaces, pointwise multipliers.

a continuous imbedding. The notation $A \approx B$, where A and B are norms, means that $C_1 A \leqslant B \leqslant C_2 B$ for some positive constants C_1, C_2.

1. General results. We consider a Banach space X of tempered distributions in $\mathbf{R}^n$, i.e. $X \subset \mathcal{S}'$.

REMARK. Sometimes we are forced to work modulo polynomials of some fixed degree, but in order not to complicate things we presently disregard this.

If $x \in \mathbf{R}^n$ and $h \in \mathbf{R}^n$ we put

$$\langle x, h\rangle = \sum_{i=1}^{n} x_i h_i \quad \text{and} \quad \chi_h(x) = e^{i\langle x, h\rangle}.$$

We are interested in subspaces of X invariant under multiplication by χ_h. There is a largest such space which we denote by $\pi \cdot X$, i.e. $f \in \pi \cdot X$ iff $f \in \mathcal{S}'$ and $\chi_h f \in X$ for each $h \in \mathbf{R}^n$. On $\pi \cdot X$ we have a natural topology given by the norms $f \to \|\chi_h f\|$ where h ranges over $\mathbf{R}^n$.

We further denote by $\pi_c X$ the space of all $f \in \mathcal{S}'$ such that $\chi_h f$ is a bounded set in X when h belongs to a compact set in $\mathbf{R}^n$. We give this space the topology defined by the norms $f \to \sup_{h \in K}\|\chi_h f\|$ where K runs over all compact subsets of $\mathbf{R}^n$.

Finally we want to put global restriction on $\|\chi_h f\|$. If ω is a given positive function on $\mathbf{R}^n$ we let $\pi_{\omega(h)} X$ be the space of all $f \in \pi \cdot X$ satisfying $\sup_{h \in \mathbf{R}^n}\|\chi_h f\|/\omega(h) < \infty$. With the norm $f \to \sup_{h \in \mathbf{R}^n}\|\chi_h f\|/\omega(h)$, $\pi_{\omega(h)} X$ becomes a Banach space. With no loss of generality we may assume that ω is submultiplicative, i.e. $\omega(h_1 + h_2) \leqslant \omega(h_1)\omega(h_2)$. However the only case of real interest for us is $\omega(h) = (1 + |h|^2)^{s/2}$, $s \geqslant 0$. Therefore we right away introduce the abbreviation $\pi_s X = \pi_{(1+|h|^2)^{s/2}} X$. Clearly we have the following chain of inclusions.

$$X \supset \pi \cdot X \supset \pi_c X \supset \pi_s X \supset \pi_{s'} X \qquad (s' \leqslant s).$$

As we will see in §3, in general we cannot expect equality here. It is also clear that, for instance, $X_1 \subset X_2 \Rightarrow \pi \cdot X_1 \subset \pi \cdot X_2$, and similarly for π_c and $\pi_{\omega(h)}$.

We now establish further properties of these spaces.

LEMMA 1. *Let G be a group of affine transformations on $\mathbf{R}^n$ which acts continuously on X. Then G acts continuously on $\pi \cdot X$. In particular, if X is translation (dilation) invariant, $\pi \cdot X$ is translation (dilation) invariant.*

PROOF. Let $a \in G$. Then we can write $ax = \tau + Ax$ where $\tau \in \mathbf{R}^n$ and A is a nonsingular linear transformation. Let $f \in \pi \cdot X$ and take $h \in \mathbf{R}^n$. Then we have the formula

$$\chi_h a(f) = \exp(-i\langle A^{-1}\tau, h\rangle) a(\chi_{(A')^{-1}h} f).$$

Therefore $\chi_h a(f) \in X$ for each $h \in \mathbf{R}^n$, i.e. $a(f) \in \pi \cdot X$. The continuity is obvious. $\square$

LEMMA 2. *Assume that X is relatively closed in $\mathcal{D}'$ in the sense of Gagliardo* **[3]** *(i.e. if $(\varphi_\nu)_{\nu \in Z}$ is a bounded sequence in X which converges to φ in $\mathcal{D}'$ then $\varphi \in X$). Let $f \in \pi_{\omega(h)} X$ and $\varphi \in \mathcal{S}$ with $\int \omega(h)|\hat{\varphi}(h)|\, dh < \infty$. Then $\varphi f \in X$.*

PROOF. For any linear combination of characters χ_{h_i} we have the inequality:

$$\left\| \sum c_i \chi_{h_i} f \right\| \leqslant \sum |c_i| \omega(h_i) \|f\|_{\pi_{\omega(h)} X}. \tag{1}$$

By Fourier's inversion formula,

$$\varphi(x) = (2\pi)^{-n} \int e^{i\langle x, h\rangle} \hat{\varphi}(h)\, dh.$$

If we approximate the integral with suitable Riemann sums we get a sequence $(\varphi_\nu)_{\nu \in Z}$ of finite linear combinations of characters which converge to φ in C^∞. It follows then that $\varphi_\nu f \to \varphi f$ in $\mathfrak{D}'$. Moreover (1) shows that $(\varphi_\nu f)_{\nu \in Z}$ is bounded in X. As X is relatively closed in $\mathfrak{D}'$ we may conclude that $\varphi f \in X$. □

REMARK. The assumptions of Lemma 2 are, in particular, fulfilled if X is a dual space. If $f \in \pi_C X$ and supp $\hat{\varphi}$ is compact the above proof also yields $\varphi f \in X$. (This will be needed in Theorem 3.)

2. Some function spaces. In this section we define some of the spaces which will be needed.

2.1. *Besov spaces* (*see* **[1]**, **[8]**). Let $(\varphi_\nu)_{\nu \in Z}$ be a family of testfunctions such that:

$\varphi_\nu \in \mathbb{S}$, supp $\hat{\varphi}_\nu \subset \{2^{\nu-1} \leqslant |\xi| \leqslant 2^{\nu+1}\}$,

$|\hat{\varphi}_\nu(\xi)| \geqslant C_\varepsilon > 0$ if $2^\nu(2-\varepsilon)^{-1} \leqslant |\xi| \leqslant 2^\nu(2-\varepsilon)$, for each $\varepsilon > 0$,

$|D^\alpha \hat{\varphi}_\nu(\xi)| \leqslant c_\alpha |\xi|^{-|\alpha|}$ for every multi-index α.

Without loss of generality we may assume that, for a suitable φ_0,

$$\varphi_\nu(x) = 2^{\nu n} \varphi_0(2^\nu x).$$

In what follows s, p, q will always denote numbers such that $s \in \mathbf{R}$, $1 \leqslant p, q \leqslant \infty$. We then define the homogeneous Besov space $\dot{B}_p^{s,q}$ to be the space of all distributions $f \in \mathbb{S}'$ such that $\|f\|_{\dot{B}_p^{s,q}} < \infty$, where

$$\|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{-\infty}^{\infty} (2^{\nu s} \|\varphi_\nu * f\|_p)^q \right)^{1/q}.$$

Further, let Φ denote a function satisfying:

$\Phi \in \mathbb{S}$, supp $\hat{\Phi} \subset \{|\xi| \leqslant 1\}$,

$|\hat{\Phi}(\xi)| \geqslant C_\varepsilon > 0$ if $|\xi| \leqslant 1 - \varepsilon$, for each $\varepsilon > 0$,

$\hat{\Phi}(\xi) = 1$ if $|\xi| \leqslant \frac{1}{2}$.

Moreover, we define Φ_ν by $\Phi_\nu(x) = 2^{\nu n} \Phi(2^\nu x)$. The inhomogeneous Besov space $B_p^{s,q}$ is now the space of all $f \in \mathbb{S}'$ such that $\|f\|_{B_p^{s,q}} < \infty$ where

$$\|f\|_{B_p^{s,q}} = \|\Phi * f\|_p + \left(\sum_{1}^{\infty} (2^{\nu s} \|\varphi_\nu * f\|_p)^q \right)^{1/q}.$$

Finally we introduce, mainly for technical reasons, the space $\mathfrak{B}_p^{s,q}$. Put

$$\|f\|_{\mathfrak{B}_p^{s,q}} = \left(\sum_{-\infty}^{\infty} (2^{\nu s} \|\Phi_\nu * f\|_p)^q \right)^{1/q}.$$

Then $\mathfrak{B}_p^{s,q}$ is the space of all $f \in \mathbb{S}'$ such that $\|f\|_{\mathfrak{B}_p^{s,q}} < \infty$.

The following proposition gives some relations between these spaces.

PROPOSITION 1. (i) $\mathcal{B}_p^{s,q} = \dot{B}_p^{s,q}$ *if* $s < 0$.
(ii) $\mathcal{B}_p^{0,\infty} = L^p$ *if* $p > 1$, $\mathcal{B}_1^{0,\infty} = \mathfrak{M}$.
(iii) $\mathcal{B}_p^{s,q} = 0$ *if* $s > 0$ *or* $s = 0$ *and* $q < \infty$.
(iv) $B_p^{s,q} = L_p \cap \dot{B}_p^{s,q}$ *if* $s > 0$.

PROOF. (i) We begin by proving $\mathcal{B}_p^{s,q} \subset \dot{B}_p^{s,q}$. We have $\varphi_\nu = \Phi_{\nu+1} * \varphi_\nu$, as is easily seen by taking Fourier transforms. Young's inequality then yields

$$\|\varphi_\nu * f\|_p \leqslant \|\varphi_\nu\|_1 \|\Phi_{\nu+1} * f\|_p \leqslant C\|\Phi_{\nu+1} * f\|_p.$$

This immediately implies that

$$\|f\|_{\dot{B}_p^{s,q}} \leqslant C\|f\|_{\mathcal{B}_p^{s,q}}.$$

Conversely we now prove that $\dot{B}_p^{s,q} \subset \mathcal{B}_p^{s,q}$ if $s < 0$. Assuming, in addition, that $\sum_{-\infty}^{\infty} \hat{\varphi}_\nu(\xi) = 1$, we obtain as above that $\Phi_\nu = \sum_{\mu<\nu+1} \varphi_\mu * \Phi_\nu$. By application of the triangle inequality and Young's inequality we obtain

$$\begin{aligned} 2^{\nu s}\|\Phi_\nu * f\|_p &\leqslant \sum_{\mu<\nu+1} 2^{\nu s}\|\Phi_\nu\|_1 \|\varphi_\mu * f\|_p \\ &\leqslant C \sum_{\lambda>-1} 2^{\lambda s}\left(2^{(\nu-\lambda)s}\|\varphi_{\nu-\lambda} * f\|_p\right). \end{aligned}$$

Minkowsky's inequality now implies that

$$\|f\|_{\mathcal{B}_p^{s,q}} \leqslant C\left(\sum_{\lambda>-1} 2^{\lambda s}\right)\|f\|_{\dot{B}_p^{s,q}},$$

where the geometrical sum converges since $s < 0$.

(ii) If $f \in \mathcal{B}_p^{0,\infty}$ we have $\sup_\nu \|\Phi_\nu * f\|_p \leqslant C$. Since $\Phi_\nu * f \to f$ in $\mathcal{S}'$ it follows by a classical argument involving weak compactness that $f \in L^p$ if $p > 1$ and $f \in \mathfrak{M}$ if $p = 1$.

(iii) If $s > 0$ we may clearly assume that $q = \infty$. For $f \in \mathcal{B}_p^{s,\infty}$ it follows that $\|\Phi_\nu * f\|_p \leqslant C2^{-\nu s}$. Thus $\Phi_\nu * f \to 0$ in L^p. But as in (ii), $\Phi_\nu * f \to f$ in $\mathcal{S}'$ and therefore $f \equiv 0$ if $s > 0$. The case $s = 0$, $q < \infty$ is handled similarly.

(iv) See [**1**, p. 148]. □

2.2. *Sobolev spaces* (*see* [**8**]). Let $k \in Z_+$ and $1 \leqslant p \leqslant \infty$. The Sobolev space W_k^p is then the space of all $f \in \mathcal{S}'$ such that $D^\alpha f \in L^p$ for $|\alpha| \leqslant k$.

2.3. *Spaces of Morrey type* (*see* [**6**], [**7**]). In what follows let $\lambda > 0$, $1 \leqslant p \leqslant \infty$. If $f \in L^1_{\text{loc}}$, $x \in \mathbf{R}^n$ and $r > 0$ we put

$$G_\lambda^p(f, x, r) = \left(r^{-\lambda}\int_{|x-y|<r} |f(y)|^p\, dy\right)^{1/p}.$$

The space $M_q^{p,\lambda}(\cdot)$ is defined to consist of all $f \in L^1_{\text{loc}}$ such that for each $x \in \mathbf{R}^n$ the norm

$$\|f\|_{M_q^{p,\lambda}(\cdot),x} = \left(\sum_{-\infty}^{\infty} \left(G_\lambda^p(f, x, 2^\nu)\right)^q\right)^{1/q}$$

is finite. We equip $M_q^{p,\lambda}(\cdot)$ with the topology given by the totality of these norms as x ranges over $\mathbf{R}^n$.

In analogy with §1 we now introduce the space $M_q^{p,\lambda}(C)$. A function $f \in L^1_{loc}$ belongs to $M_q^{p,\lambda}(C)$ iff the norms

$$\|f\|_{M_q^{p,\lambda}(C),K} = \left(\sum_{-\infty}^{\infty} \left(\sup_{x \in K} G_\lambda^p(f, x, 2^\nu) \right)^q \right)^{1/q}$$

are finite for every compact subset K in $\mathbf{R}^n$. Using these norms we define a topology on $M_q^{p,\lambda}(C)$.

Let ω be a given positive function on $\mathbf{R}^n$. The space $M_q^{p,\lambda}(\omega(x))$ then consists of all $f \in L^1_{loc}$ satisfying

$$\|f\|_{M_q^{p,\lambda}(\omega(x))} = \left(\sum_{-\infty}^{\infty} \left(\sup_{x \in \mathbf{R}^n} G_\lambda^p(f, x, 2^\nu)/\omega(x) \right)^q \right)^{1/q} < \infty.$$

With this norm $M_q^{p,\lambda}(\omega(x))$ will be a Banach space. In particular, if $\omega(x) = (1 + |x|^2)^{s/2}$, $s > 0$, which is the only case of interest for us, we write $M_q^{p,\lambda}(s)$ instead of $M_q^{p,\lambda}((1 + |x|^2)^{s/2})$. The usual Morrey spaces correspond to $M_q^{p,\lambda}(0)$ in our notation. They agree with the Stampacchia spaces $L_q^{p,\lambda}$ if $0 < \lambda < n$, $q < \infty$ or $0 < \lambda \leq n$, $q = \infty$. See [2] and [7]. We clearly have the following inclusions.

$$M_q^{p,\lambda}(\cdot) \supset M_q^{p,\lambda}(C) \supset M_q^{p,\lambda}(s).$$

PROPOSITION 2. (i) $M_q^{p,\lambda}(\cdot) = 0$ *if* $\lambda > n$ *or* $\lambda = n$ *and* $q < \infty$, *or* $\lambda = 0$ *and* $q < \infty$.

(ii) $M_\infty^{p,n}(0) = L^\infty$.

(iii) $M_\infty^{p,0}(\cdot) = M_\infty^{p,0}(0) = L^p$.

PROOF. (i) If $\lambda > n$ we may assume that $q = \infty$. Take $f \in M_\infty^{p,\lambda}(\cdot)$ and let x be a Lebesgue point for f. For some C we have

$$\frac{1}{r^n} \int_{|x-y|<r} |f(y)|^p \, dy \leq C r^{\lambda - n}. \tag{2}$$

Lebesgue's theorem then implies that the left side approaches $|f(x)|$ as $r \to 0$. Thus $f(x) \equiv 0$ a.e. if $\lambda > n$.

The case $\lambda = n$, $q < \infty$ is treated similarly. With x a Lebesgue point as before, then for some ν_0 we must have

$$\left(\frac{1}{2^{\nu n}} \int_{|x-y|<2^\nu} |f(y)|^p \, dy \right)^{q/p} \geq \tfrac{1}{2} |f(x)|^q$$

if $\nu \leq \nu_0$. Summing over ν we see that f must vanish a.e. The case $\lambda = 0$, $q < \infty$ is trivial.

(ii) If $f \in M_\infty^{p,n}(0)$ we may choose C in (2) independent of x. Letting $\nu \to -\infty$ we see that $f \in L^\infty$, again by Lebesgue's theorem.

(iii) Trivial. □

3. Determination of $\pi \cdot \dot{B}_p^{s,q}$. We now turn to the main topic of this paper. Since the three cases $s > 0$, $s = 0$ and $s < 0$ behave quite differently, we divide this section into three parts.

3.1. *The case* $s > 0$.

THEOREM 1. *If $s > 0$ and $1 \leqslant p, q \leqslant \infty$ then $\pi \cdot \dot{B}_p^{s,q} = B_p^{s,q}$. Moreover we have for each $h \in \mathbf{R}^n$ and f in this space*

$$\|f\|_{\dot{B}_p^{s,q}} + \|\chi_h f\|_{\dot{B}_p^{s,q}} \approx |h|^s \|f\|_p + \|f\|_{\dot{B}_p^{s,q}}. \tag{3}$$

The main component in the proof will be the observation that multiplication by χ_h corresponds to translation on the Fourierside, which is expressed in the following lemma.

LEMMA 3. *Assume that $\psi \in L^1$ and* supp $\hat{\psi}$ *is compact. For some ν_0 we then have the inequality*

$$\|\psi * f\|_p \leqslant \|\psi\|_1 \|\varphi_\nu * \chi_h f\|_p$$

if $\nu \geqslant \nu_0$, $|h| \approx 2^\nu$ and the right-hand side is finite, $1 \leqslant p \leqslant \infty$.

PROOF OF LEMMA 3. Choose $(\varphi_\nu)_{\nu \in Z}$, as in the definition of $\dot{B}_p^{s,q}$, satisfying, in addition, $\hat{\varphi}_\nu(\xi) = 1$ if $\xi \in I_\nu = \{\xi\colon \mid |\xi| - 2^\nu \mid \leqslant C_0 2^\nu\}$. As supp $\hat{\psi}$ is compact, there exists a ν_0 such that supp $\hat{\psi} \subset \{|\xi| \leqslant C_0 2^{\nu_0}\}$. Thus if $\nu \geqslant \nu_0$, $|h| \approx 2^\nu$, we have $\operatorname{supp}(\chi_h \psi)\hat{} = h + \operatorname{supp} \hat{\psi} \subset I_\nu$. Our choice of $(\varphi_\nu)_{\nu \in Z}$ then implies that $\chi_h \psi * \varphi_\nu = \chi_h \psi$. Rewriting $\chi_h(\psi * f)$ as $\varphi_\nu * \chi_h \psi * \chi_h f$ and applying Young's inequality, we get

$$\|\psi * f\|_p = \|\chi_h(\psi * f)\|_p \leqslant \|\psi\|_1 \|\varphi_\nu * \chi_h f\|_p.$$

PROOF OF THEOREM 1. We begin by proving (3) for a fixed h. With $\psi = \Phi$, as in the definition of $B_p^{s,q}$, and h_0 such that $|h_0| \approx 1$ we obtain from Lemma 3,

$$\|\Phi * f\|_p \leqslant C \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}}.$$

This implies that

$$\|f\|_{B_p^{s,q}} \leqslant C \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}}. \tag{4}$$

In order to obtain the converse inequality we first use the imbedding $B_p^{s,q} \subset \dot{B}_p^{s,q}$, $s > 0$ (Proposition 1(iv)) which gives

$$\|f\|_{\dot{B}_p^{s,q}} \leqslant C \|f\|_{B_p^{s,q}},$$
$$\|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} \leqslant C \|\chi_{h_0} f\|_{B_p^{s,q}}. \tag{5}$$

But χ_h acts continuously on $B_p^{s,q}$. Indeed, in view of the fact that $B_p^{s,q} = (L^p, W_k^p)_{\theta,q}$, $s = \theta k$, $0 < \theta < 1$ (see [**8**, p. 64]), it suffices to show that χ_h acts continuously on W_k^p, which is obvious. We thus have

$$\|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} \leqslant C \|f\|_{B_p^{s,q}}. \tag{6}$$

By combining (4)–(6) we get the desired inequalities:

$$\|f\|_{B_p^{s,q}} \approx \|\chi_{h_0} f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}} \quad \text{with } |h_0| \approx 1. \tag{7}$$

An argument with dilations will now establish (3). Indeed take $0 \neq h \in \mathbf{R}^n$. As rotations act continuously on $\dot{B}_p^{s,q}$ (see Lemma 1) we may assume that h and h_0 are collinear, i.e. $h = \lambda h_0$ for some $\lambda > 0$.

Let τ_δ denote the dilation operator defined by

$$(\tau_\delta f)(x) = f(\delta x), \qquad \delta > 0.$$

It is well known that τ_δ acts continuously on $\dot{B}_p^{s,q}$ and we have

$$\|\tau_\delta f\|_{\dot{B}_p^{s,q}} \approx \delta^{s-n/p}\|f\|_{\dot{B}_p^{s,q}}. \tag{8}$$

We further notice the formula

$$\chi_h f = \tau_\lambda(\chi_{h_0}\tau_{1/\lambda} f). \tag{9}$$

If we apply (7) with $\tau_{1/\lambda} f$ and multiply by $\lambda^{s-n/p}$ we get, in view of (8) and (9),

$$\|\chi_h f\|_{\dot{B}_p^{s,q}} + \|f\|_{\dot{B}_p^{s,q}} \approx \lambda^{s-n/p}\|\tau_{1/\lambda} f\|_{B_p^{s,q}}.$$

Finally, the right side is simplified by once again invoking Proposition 1(iv) thereby obtaining

$$\begin{aligned}\lambda^{s-n/p}\|\tau_{1/\lambda} f\|_{B_p^{s,q}} &\approx \lambda^{s-n/p}\left(\|\tau_{1/\lambda} f\|_p + \|\tau_{1/\lambda} f\|_{\dot{B}_p^{s,q}}\right)\\ &\approx \lambda^s\|f\|_p + \|f\|_{\dot{B}_p^{s,q}}.\end{aligned}$$

As $\lambda^s \approx |h|^s$ we have now proved (3), and thus $\pi \cdot \dot{B}_p^{s,q} = B_p^{s,q}$. □

COROLLARY 1. *Let $s > 0$, $1 \leqslant p, q \leqslant \infty$. Then $\pi_{s'}\dot{B}_p^{s,q} = B_p^{s,q}$ if $s' \geqslant s$, $\pi_{s'}\dot{B}_p^{s,q} = 0$ if $s' < s$.*

2.2. *The case $s = 0$.*

THEOREM 2. (i) *If $1 \leqslant p, q \leqslant \infty$ then $\pi \cdot \dot{B}_p^{0,q} \subset B_p^{0,q}$.*
(ii) *If $q = \infty$ and $1 \leqslant p \leqslant \infty$ or $q \geqslant \max(2, p)$ and $1 < p < \infty$, then*

$$\pi_0\dot{B}_p^{0,q} = L^p \quad \text{if } p > 1, \ \pi_0\dot{B}_1^{0,\infty} = \mathfrak{M}.$$

PROOF. (i) As in the proof of Theorem 1 we obtain

$$\|f\|_{B_p^{0,q}} \leqslant C\|\chi_h f\|_{\dot{B}_p^{0,q}} + \|f\|_{\dot{B}_p^{0,q}} \quad \text{if } |h| \approx 1.$$

Thus $\pi \cdot \dot{B}_p^{0,q} \subset B_p^{0,q}$.

(ii) If p and q are as in the hypothesis, we have the imbedding $L^p \subset \dot{B}_p^{0,q}$ (see [**8**, p. 80]). It follows that $L^p \subset \pi_0\dot{B}_p^{0,q}$. We now show that, conversely, $\pi_0\dot{B}_p^{0,q} \subset L^p$ holds. Take $f \in \pi_0\dot{B}_p^{0,q}$. Then for each ν, $\Phi_\nu * f \in \pi_0\dot{B}_p^{0,q}$. Lemma 3 now gives, if $|h| \approx 2^\nu$ and $\psi = \Phi_{\nu+1}$,

$$\begin{aligned}\|\Phi_\nu * f\|_p &\leqslant C\|\chi_h(\Phi_\nu * f)\|_{\dot{B}_p^{0,q}}\\ &\leqslant C\|\Phi_\nu * f\|_{\pi_0\dot{B}_p^{0,q}}\\ &\leqslant C\|f\|_{\pi_0\dot{B}_p^{0,q}},\end{aligned}$$

so by Proposition 1(ii), $f \in L^p$ if $p > 1$. $f \in \mathfrak{M}$ if $p = 1$. □

2.3. *The case $s < 0$.* We first observe that Proposition 1(ii) allows us to replace $\dot{B}_p^{s,q}$ with $\mathfrak{B}_p^{s,q}$ as $s < 0$. This will be done in the proofs given below. It will be convenient to introduce the following terminology.

DEFINITION. We say that s, p, q, where $s < 0$, $1 \leqslant p, q \leqslant \infty$, are good indices iff $-n/p' \leqslant s < 0$ if $q = \infty$, or $-n/p' < s < 0$ if $q < \infty$. (p' denotes the conjugate index to p.) Otherwise we say that s, p, q are bad indices.

Using this notation we now state our main result.

THEOREM 3. *If s, p, q are bad indices then*
(i) $\pi \cdot \dot{B}_p^{s,q} \cap L^1 = 0$,
(ii) $\pi_c \dot{B}_p^{s,q} = 0$.

PROOF. We reduce to the case $p = \infty$. This may be accomplished by using Besov's imbedding theorem (see [8, p. 63]) implying that $\dot{B}_p^{s,q} \subset \dot{B}_\infty^{s-n/p,q}$.

(i) Take $f \in \pi \cdot \dot{B}_\infty^{s,q} \cap L^1$. We then obtain, using Parseval's formula,

$$2^{\nu s}\|\Phi_\nu * f\|_\infty \geqslant 2^{\nu s}|(\Phi_\nu * f)(0)| = 2^{\nu s}\left|\int \Phi_\nu(x)f(-x)\,dx\right|$$

$$= C2^{\nu s}\left|\int \hat{\Phi}(\eta/2^\nu)\hat{f}(-\eta)\,d\eta\right|.$$

We may assume that $\hat{\Phi}(\eta) \geqslant 0$. If we apply the above inequality to $\chi_h(f * \bar{f})$ and then use Young's inequality we find that

$$2^{\nu s}\int_{|\eta - h| < 2^{\nu-1}} |\hat{f}(\eta)|^2\,d\eta \leqslant C2^{\nu s}\|\Phi_\nu * \chi_h f\|_\infty \|f\|_1.$$

Therefore it follows that $\hat{f} \in M_{2q}^{2,-s}(\cdot)$. After invoking Proposition 2(ii), we arrive at the result $\hat{f} \equiv 0$ if s, ∞, q are bad indices.

(ii). Take $f \in \pi_c \dot{B}_\infty^{s,q}$. Our aim is to modify f so that we may use (i). Assume first that $q > 1$. Take ψ with $\hat{\psi} \in C_0^\infty$. Then $f * \psi \in \pi_c \dot{B}_\infty^{s,q}$. As $\operatorname{supp}(\psi * f)\hat{}$ is compact, it is easily seen that $f * \psi \in L^\infty$. Indeed we clearly have for some finite N,

$$f * \psi = \sum_{-\infty}^{N} \varphi_\nu * f * \psi.$$

As $s < 0$, an application of Minkowsky's inequality yields

$$\|f * \psi\|_\infty \leqslant \|f * \psi\|_{\dot{B}_\infty^{0,1}} \leqslant 2^{-Ns}\|f * \psi\|_{\dot{B}_\infty^{s,q}}.$$

Take $\varphi \in L^1 \cap \mathcal{S}$ with supp $\hat{\varphi}$ compact. Hölder's inequality now implies that $\varphi(f * \psi) \in L^1$. As $\dot{B}_\infty^{s,q}$ is a dual space for $q > 1$ (see [8, p. 74]), we find, by using Lemma 2, that $\varphi(f * \psi) \in \dot{B}_\infty^{s,q}$. Thus $\varphi(f * \psi) \in \pi \cdot \dot{B}_\infty^{s,q} \cap L^1$ if we apply the above argument to $\chi_h \varphi(f * \psi)$. (i) then shows that $\varphi(f * \psi) \equiv 0$ for all φ and ψ. Thus f must vanish.

The case $q = 1$ follows trivially from what we have proved for $q > 1$. Just notice that $\dot{B}_p^{s,1} \subset \dot{B}_p^{s,q}$ for any $q \geqslant 1$. □

The main idea in this proof was the pointwise behaviour of f. The Hausdorff-Young theorem allows us to sharpen Theorem 2 if $1 < p < 2$. Let $\mathfrak{F}$ denote the Fourier transform.

THEOREM 4. *Let $s < 0$, $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \infty$. Then $\mathfrak{F}: \pi \cdot \dot{B}_p^{s,q} \to M_q^{p',-sp'}(\cdot)$. In particular, $\pi \cdot \dot{B}_p^{s,q} = 0$ if s, p, q are bad indices.*

PROOF. An application of the Hausdorff-Young theorem yields for each $h \in \mathbf{R}^n$ and ν,

$$2^{\nu s}\|\Phi_\nu * \chi_h f\|_p \geqslant C2^{\nu s}\|\hat{\Phi}(\cdot/2^\nu)\hat{f}(\cdot - h)\|_{p'}$$

$$\geqslant C\left(2^{\nu s p'}\int_{|\eta - h| < 2^{\nu-1}} |\hat{f}(\eta)|^{p'}\,d\eta\right)^{1/p'}$$

$$= CG_{-sp'}^{p'}(\hat{f}, h, 2^{\nu-1}).$$

Raising this to the qth power and summing over ν we obtain

$$\|\hat{f}\|_{M_q^{p',-sp'}(\cdot),h} \leqslant C\|\chi_h f\|_{\dot{B}_p^{s,q}}.$$

This proves the first part of the theorem.

That $\pi \cdot \dot{B}_p^{s,q} = 0$, for s, p, q bad indices, is now a consequence of Proposition 2(i). □

Conversely, reversing all inequality signs in the above proof, we obtain the following theorem and corollary.

THEOREM 5. *If* $s < 0$, $2 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. *Then* $\mathcal{F}\colon M_q^{p',-sp'}(\cdot) \to \pi \cdot \dot{B}_p^{s,q}$.

COROLLARY 2. *If* $s < 0$, $1 \leqslant q \leqslant \infty$ *then*

$$\pi \cdot \dot{B}_2^{s,q} = \mathcal{F} M_q^{2,-2s}(\cdot).$$

REMARKS. (i) Theorems 4 and 5 and Corollary 2 have obvious extensions to the spaces $\pi_{\omega(h)}\dot{B}_p^{s,q}$ and $\pi_c\dot{B}_p^{s,q}$. The conclusions will then involve the spaces $M_q^{p,\lambda}(\omega(x))$ and $M_q^{p,\lambda}(C)$.

(ii) Corollary 2 shows that the spaces $\pi \cdot X$, $\pi_c X$ and $\pi_{\omega(h)}X$ do not coincide in general. For instance it is easily seen that $M_q^{2,-2s}(\cdot)$ and $M_q^{2,-2s}(C)$ behave topologically differently. Further, $f \in M_\infty^{2,n}(\omega(x))$ implies that $|f(x)| < C\omega(x)$. As $M_\infty^{2,n}(0) = L^\infty$ we thus see that $\pi_{\omega(h)}\dot{B}_2^{-n/2,\infty} \neq \pi_0\dot{B}_2^{-n/2,\infty}$ if, e.g., $\omega(h) \to 0$ as $h \to \infty$.

As is seen by Theorem 5, $\pi \cdot \dot{B}_p^{s,q}$ is a "large" space if $2 < p < \infty$ and s, p, q are good indices. This is true without any restrictions on p, as is seen from the following proposition.

PROPOSITION 3. $\mathcal{S} \subset \pi_0\dot{B}_p^{s,q}$ *if* s, p, q *are good indices.*

We omit the proof since this simply amounts to a use of Young's inequality.

REMARK. Some of our results are already found in [**4**]. More precisely, Johnson considers $\pi_s\dot{B}_p^{s,q}$ if $s > 0$ and $\pi_0\dot{B}_p^{s,\infty}$ if $s \leqslant 0$. He establishes a weaker version of Theorem 1. He also proves Theorem 2(ii), if $q = \infty$, Theorems 4, 5 and Corollary 2 for $\pi_0\dot{B}_p^{s,\infty}$.

REFERENCES

1. J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer, Berlin and New York, 1976.

2. S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 137–160.

3. E. Gagliardo, *A unified structure in various families of function spaces. Compactness and closure theorems*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.

4. R. Johnson, *Maximal subspaces of Besov spaces invariant under multiplication by characters*, Trans. Amer. Math. Soc. **249** (1979), 387–407.

5. ______, *Duality methods for the study of maximal subspaces of Besov spaces invariant under multiplication by characters*, Univ. of Maryland Tech. Report 79, 75.

6. C. B. Morrey, *Functions of several variables and absolute continuity*, Duke Math. J. **6** (1940), 187–215.

7. J. Peetre, *On the theory of $L_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.

8. ______, *New thoughts on Besov spaces*, Duke Univ. Math. Ser. Durham, N. C., 1976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LUND, S-22007 LUND, SWEDEN